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COMPLETION OF RESTRICTED LIE ALGEBRAS

BY

D. M. RILEY* AND J. F. SEMPLE

Mathematical Institute, 24–29 St. Giles', Oxford OX1 3LB, England

ABSTRACT

We study pro-'finite dimensional finite exponent' completions of restricted Lie algebras over finite fields of characteristic p. These compact Hausdorff topological restricted Lie algebras, called *pro-$\mathcal{F}_p$ restricted Lie algebras*, are the restricted Lie-theoretic analogues of pro-p groups. A structure theory for pro-$\mathcal{F}_p$ restricted Lie algebras with finite rank is developed. In particular, the centre of such a Lie algebra is shown to be open. As an application we examine p-adic analytic pro-p groups in terms of their associated pro-$\mathcal{F}_p$ restricted Lie algebras.

1. Introduction

We propose to study a class of topological restricted Lie algebras which are the analogues of pro-p groups. Consider the class $\mathcal{F}_p$ of all finite dimensional restricted Lie algebras over a finite field of characteristic p whose p-map is nilpotent. A restricted ideal I of a restricted Lie algebra L is said to be an **$\mathcal{F}_p$-ideal** if L/I lies in the class $\mathcal{F}_p$. A **pro-$\mathcal{F}_p$ restricted Lie algebra** is a compact Hausdorff topological restricted Lie algebra L whose open $\mathcal{F}_p$-ideals form a neighbourhood base of 0. For the sake of brevity, we refer to pro-$\mathcal{F}_p$ restricted Lie algebras as **pro-p algebras** hereafter. We shall see that the analogy between pro-p groups and pro-p algebras often bears close scrutiny. The reader is referred to the recent monograph of Dixon, du Sautoy, Mann and Segal, [DDMS], for an exposition on pro-p groups and several of their applications to abstract group theory.

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The primary focus of our investigation will be to determine the structure of topologically finitely generated pro-p algebras. Specifically, we are interested in pro-p algebras with finite rank. For our purposes, the **rank** of a topological algebra A is the minimum number r such that every closed finitely generated subalgebra of A can be topologically generated by r elements. If $r < \infty$ then A is said to have **finite rank**. Our main results are somewhat reminiscent of those due to Lubotzky and Mann, [LM2], concerning pro-p groups of finite rank. We show that a finitely generated pro-p algebra L has finite rank precisely when its centre is open. Moreover, in this case L contains a central restricted subalgebra U which has the structure of a finitely generated free abelian pro-p algebra. The rank of U is an invariant of L, which we shall refer to as the **uniform dimension** of L.

Before proving these results, we examine pro-p algebras more generally. First we use a theorem of Witt's to demonstrate that an open restricted subalgebra of a finitely generated pro-p algebra is finitely generated. Continuing in the lines of Lincoln and Towers, [LT], we then characterise the Frattini subalgebra of a pro-p algebra, which in turn we use to show that the topology of a finitely generated pro-p algebra is uniquely determined by its algebraic structure. Subsequently, we are able to reduce the study of pro-p algebras with finite rank to the study **powerful** pro-p algebras, which are nilpotent of class at most two.

In the concluding section we apply our theory to finitely generated pro-p groups. Let $\mathcal{L}(G) = \prod_{m\geq 1} D_m(G)/D_{m+1}(G)$, where $D_m(G)$ is the mth dimension subgroup of G over $\mathbb{F}_p$. Then $\mathcal{L}(G)$ is the **associated pro-p algebra** of G. We show, for example, that G is p-adic analytic if and only if $\mathcal{L}(G)$ has finite rank. Moreover, in this case the uniform dimension of $\mathcal{L}(G)$ equals the dimension of G. These results are then employed to obtain a quantitative version of a theorem due to Lazard, [L2], and Shalev, [S], who proved that a finitely generated pro-p group G has finite rank if and only if $\mathcal{L}(G)$ is nilpotent.

2. Definitions

Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. We shall assume throughout that $\mathbb{F}$ is finite. Denote the Lie product by $(x, y) \mapsto [x, y]$. Recall that L is a **restricted Lie algebra**, or **Lie p-algebra**, if it affords a unary operation $x \mapsto x^p$ satisfying

1. $(\lambda x)^p = \lambda^p x^p$ when $\lambda \in \mathbb{F}, x \in L$;

2. $\operatorname{ad} x^p = (\operatorname{ad} x)^p$; and
3. $(x+y)^p = x^p + y^p + \sum_i s_i(x,y)$,

where $is_i(x,y)$ is the coefficient of t^{i-1} in $\operatorname{ad}(tx+y)^{p-1}(x)$. Jacobson provides an introduction to Lie p-algebras in his book [J]. We say that L is a **topological Lie p-algebra** if it is also a topological space such that the operations of addition, scalar multiplication, Lie multiplication, and exponentiation by p are all continuous. See van der Waerden, [vdW], for an introduction to topological algebras.

Following Strade and Farnsteiner, [SF], a subalgebra H of L is called a **p-subalgebra** if H is closed under the p-map. We say that a Lie p-algebra is **generated** by a subset X if the smallest p-subalgebra containing X is L. Let $L^{\{p^k\}}$ denote the set of p^kth powers of elements of L, and let L^{p^k} be the p-subalgebra it generates. We write $\zeta(L)$ for the centre of L. The left-normed convention is used for longer commutators: inductively $[x_0, \ldots, x_{n-1}, x_n] = [[x_0, \ldots, x_{n-1}], x_n]$. For $n \geq 2$ we set $[x,_n y] = [x,_{n-1} y, y]$. As usual, we let $\gamma_i(L)$ denote the ith term of the lower central series of L. We often write L' for $\gamma_2(L)$. Finally, open and closed p-subalgebras are denoted by $\leq_o$ and $\leq_c$, respectively.

Definition 2.1: A Lie p-algebra L is said to be **p-nilpotent** if there is a positive integer k such that $L^{p^k} = 0$. If k is minimal then p^k is the **exponent** of L. Then $\mathcal{F}_p$ is the class of all p-nilpotent finite dimensional Lie p-algebras over $\mathbb{F}$. Recall that a p-ideal I of L is an **$\mathcal{F}_p$-ideal** if L/I lies in the class $\mathcal{F}_p$. ∎

Notice that Engel's Theorem implies that L is nilpotent if it lies in $\mathcal{F}_p$.

Definition 2.2: Let Λ be a directed set of $\mathcal{F}_p$-ideals of L, ordered by reverse inclusion. Then $\{L/I\}_{I\in\Lambda}$, together with the natural epimorphisms, forms an inverse system over Λ. The **pro-$\mathcal{F}_p$ completion of L with respect to** Λ is the topological Lie p-algebra given by

$$\hat{L} = \varprojlim (L/I)_{I\in\Lambda}.$$

The topology imposed on $\hat{L}$ is that induced by the discrete topology on the factors. ∎

Recall from Section 1 the definition of a pro-p algebra:

Definition 2.3: A **pro-p algebra** is a compact Hausdorff topological Lie p-algebra whose open $\mathcal{F}_p$-ideals form a neighbourhood base for 0. ∎

In fact, these last two definitions give rise to the same class of topological Lie p-algebras: the pro-$\mathcal{F}_p$ completion of a Lie p-algebra is a pro-p algebra, and vice versa. This follows from standard topological arguments which are identical to those used for pro-p groups. See [DDMS, pp. 21–22] for details. It is noteworthy that the field $\mathbb{F}$ is required to be finite precisely so that the underlying topology will be compact.

In the next section we shall study some elementary consequences of the definitions, but first let us consider the prototype for pro-p algebras.

Example 2.4: Let $A = \langle x \rangle$ denote the free 1-generated Lie p-algebra over $\mathbb{F}_p$, and let $\hat{A}$ denote its pro-$\mathcal{F}_p$ completion over the set of all $\mathcal{F}_p$-ideals. Then $\hat{A} \cong \mathbb{F}_p[[t]]$ where $\mathbb{F}_p[[t]]$, the set of formal power series over $\mathbb{F}_p$, is viewed as a topological abelian Lie p-algebra with p-map induced by multiplication by t and topology induced by the valuation $\mu : \mathbb{F}_p[[t]] \to \mathbb{Q}$ given by

$$\mu(\sum a_i t^i) = \begin{cases} 0, & \text{if } a_i = 0 \text{ for all } i; \\ p^{-j}, & j = \min\{i \mid a_i \neq 0\} \text{ otherwise.} \end{cases} \qquad \blacksquare$$

Indeed, let us write $A = \bigoplus_{k \geq 0} \mathbb{F}_p x^{p^k}$. It is not difficult to see that the $\mathcal{F}_p$-ideals of A are precisely the sets of the form $I_m = \bigoplus_{k \geq m} \mathbb{F}_p x^{p^k}$; therefore, if $\hat{A} = \varprojlim A/I_m$ then we may express each element $\alpha \in \hat{A}$ uniquely in the form $\alpha = \left(\alpha_i(x)\right)_{i \geq 0}$, where $\alpha_i(x) = a_0 x + a_1 x^p + \cdots + a_i x^{p^i}$. Consider the map φ: $\hat{A} \to \mathbb{F}_p[[t]]$ with $\varphi(\alpha) = \sum a_i t^i$. Then φ is an isomorphism of topological Lie p-algebras. This follows from the fact that $a \in \mathbb{F}_p$ implies that $a^p = a$, so

$$(\sum a_i x^{p^i})^p = \sum a_i^p x^{p^{i+1}} = \sum a_i x^{p^{i+1}}.$$

We shall see that $\hat{A}$ plays a role, in positive characteristic, similar to that played by the p-adic integers in the study of pro-p groups.

3. Elementary properties

In this section we shall outline some elementary properties of pro-p algebras. The proofs of the first two propositions follow easily from the definitions.

PROPOSITION 3.1: *Let L be a pro-p algebra.*

1. *A p-subalgebra of L is open if and only if it is closed and of finite codimension.*
2. *The intersection of all open p-subalgebras is $\{0\}$.*

3. *Let $\overline{X}$ denote the closure of a subset X of L. Then $\overline{X} = \bigcap_{I \triangleleft_o L}(X + I)$. Moreover, every closed p-subalgebra H is the intersection of all open p-subalgebras containing it.*
4. *If X and Y are closed subsets and k a positive integer then the sets $X+Y$, $\{[x,y] \mid x \in X, y \in Y\}$ and $\{x^{p^k} \mid x \in X\}$ are closed.*
5. *Let H be a closed p-subalgebra of L. Then H with the subspace topology is a pro-p algebra, and every open p-subalgebra of H is of the form $H \cap K$ with K an open p-subalgebra of L.*
6. *Let I be a closed p-ideal of L. Then L/I is a pro-p algebra with topology induced by the natural homomorphism.*
7. *We say that a sequence (h_i) in L is* **Cauchy** *if for every $I \triangleleft_o L$ there exists an n such that $h_i - h_j \in I$ whenever $i \geq j \geq n$. A sequence in L converges if and only if it is Cauchy.*

Definition 3.2: Let L be a topological Lie p-algebra and $X \subseteq L$. Denote by $\langle X \rangle$ the p-subalgebra of L generated by X. Then X **generates** L **(topologically)** if $\overline{\langle X \rangle} = L$. ∎

PROPOSITION 3.3: *Let L be a pro-p algebra and $X \subseteq L$.*
1. *X generates L if and only if $(\langle X \rangle + I)/I = L/I$ for every $I \triangleleft_o L$.*
2. *L is d-generated if and only if L/I is d-generated for every $I \triangleleft_o L$.*

We may now establish the pro-p algebraic analogue of a theorem due to Witt, who proved that a p-subalgebra of codimension $r < \infty$ in an n-generated Lie p-algebra is generated by $m = p^r(n-1)+1$ elements. See for example Bahturin, [B, pp. 68–69], for a proof.

THEOREM 3.4: *Let L be an n-generated pro-p algebra and let H be an open p-subalgebra of codimension r. Then H is $p^r(n-1)+1$-generated.*

Proof: For all $I \triangleleft_o L$, L/I is n-generated and $H + I/I$ is of codimension at most r in L/I. Therefore by Witt's theorem $H + I/I$ is $p^r(n-1)+1$-generated for each I. Thus, by Proposition 3.3, H is $p^r(n-1)+1$-generated. ∎

In light of this result, we may define the rank of a pro-p algebra as follows.

Definition 3.5: Let L be a pro-p algebra, and let $d(L)$ denote the minimal cardinality of a generating set of L. If there exists an integer r such that $d(H) \leq r$

for every open p-subalgebra H of L, then the least such r is the called the **rank** of L; otherwise L is said to be of infinite rank. We denote r by $\mathrm{rk}(L)$. ∎

Observe that L is finitely generated if it has finite rank. Equivalent definitions of rank are given in Proposition 6.1.

4. Frattini theory for pro-p algebras

Presently we examine the Frattini p-subalgebra of a pro-p algebra. It transpires that this subalgebra is a closed p-ideal which is open precisely when L is finitely generated.

Definition 4.1: Let L be a pro-p algebra. The **Frattini p-subalgebra** $\Phi(L)$ is defined to be the intersection of all maximal open p-subalgebras of L. ∎

LEMMA 4.2: *Let L be a pro-p algebra.*

1. *If $X + \Phi(L)$ generates L then X generates L.*
2. $d(L) = \dim L/\Phi(L)$.

The proof is standard. See Amayo and Stewart, [AS], pp. 241–242.

COROLLARY 4.3: *A pro-p algebra L is finitely generated if and only if $\Phi(L)$ is open in L.*

Proof: Assume that $\Phi(L)$ is open, so that a vector space complement X of $\Phi(L)$ in L is a finite set. Because the Frattini p-ideal is the set of nongenerators of L, then $L = \overline{\langle X\rangle}$. Conversely, notice that $\Phi(L)$ is closed because it is the intersection of open p-subalgebras. Therefore if $d(L) < \infty$ then $\Phi(L)$ is open by Proposition 3.1 and Lemma 4.2. ∎

LEMMA 4.4: *Let L be a pro-p algebra. Then $\Phi(L) = \overline{L^p + L'}$.*

Proof: In [LT], Lincoln and Towers established this result in the case when L is $\mathcal{F}_p$. For the general case, assume that I is an open p-ideal of L. Because $\Phi(L/I) = \Phi(L) + I/I$, we have $\Phi(L) + I = L^p + L' + I$ for all such I. Hence

$$\Phi(L) = \overline{\Phi(L)} = \bigcap_{I \triangleleft_o L} (\Phi(L) + I) = \bigcap_{I \triangleleft_o L} (L^p + L' + I) = \overline{L^p + L'}. \qquad ∎$$

COROLLARY 4.5: *If L is a finitely generated pro-p algebra then L' is closed and $\Phi(L) = L^p + L'$.*

Proof: Suppose that $\{a_1, \ldots, a_d\}$ generates L and let $I \triangleleft_o L$. Then

$$L' + I/I = (L/I)' = \sum_{i=1}^{d} [a_i, L] + I/I.$$

Therefore $\overline{L'} = \sum_{i=1}^{d} [a_i, L] = L'$ since each $[a_i, L] = (\mathrm{ad}\, a_i)L$ is closed. Now $\Phi(L) = \overline{L^p + L'} = \overline{L^{\{p\}} + L'} = L^{\{p\}} + L'$. ∎

We now establish a useful result which implies that the topology on a finitely generated pro-p algebra is uniquely determined by its algebraic structure.

THEOREM 4.6: *Let L be a finitely generated pro-p algebra and I any $\mathcal{F}_p$-ideal. Then I is open.*

Proof: We may assume by induction on $\dim L/I$ that I is open in J if J is any finitely generated pro-p algebra with $I \leq J < L$. Therefore it suffices to construct some proper open p-subalgebra J of L containing I. Put $J = \Phi(L) + I$. Then J is open in L because $\Phi(L)$ is open. Because L/I is $\mathcal{F}_p$, by Corollary 4.5 we have

$$J/I = L^p + L' + I/I = \Phi(L/I) < L/I.$$

Thus $J < L$. Also, J is a pro-p algebra by Proposition 3.1 and finitely generated by Theorem 3.4. Hence I is open in L. ∎

We now describe a series of subalgebras that will be useful for our purposes. The nth **dimension subalgebra** of an arbitrary Lie p-algebra L is defined by

$$D_n(L) = \sum_{ip^j \geq n} \gamma_i(L)^{p^j}.$$

These p-ideals arise naturally in connection with the restricted universal enveloping algebra of L: see Section 8 and [RS]. Two simple consequences of the definition of the $D_n(L)$ are that $[D_m(L), D_n(L)] \subseteq \gamma_{m+n}(L)$ and $D_m(L)^p \subseteq D_{mp}(L)$ for each pair of positive integers m, n.

COROLLARY 4.7: *Let L be a finitely generated pro-p algebra. Then $\Phi(L) = D_2(L)$ and the set of all $D_n(L)$ forms a neighbourhood base of 0.*

Proof: The first part of the claim follows from Corollary 4.5. Since $L/D_n(L)$ is finitely generated, nilpotent and of finite exponent, $L/D_n(L)$ is finite dimensional.

Thus each $D_n(L)$ is an $\mathcal{F}_p$-ideal, and hence open by Theorem 4.6. Now let I be an arbitrary open ideal of L. Since $L/I \in \mathcal{F}_p$, it is clear that $D_n(L) + I/I = D_n(L/I) = 0$ for all sufficiently large n. Therefore $D_n(L) \subseteq I$, and the second half of the claim now follows. ∎

Before closing this section, let us remark that if L is an arbitrary Lie p-algebra satisfying $\dim L/D_2(L) < \infty$ and $\hat{L}$ is any pro-$\mathcal{F}_p$ completion of L then

$$\hat{L} \cong \varprojlim_n L/D_n(L).$$

5. Powerful pro-p algebras

Let us now consider the notion of a powerful pro-p algebra. We shall see that these pro-p algebras satisfy the property $\mathrm{rk}(L) = d(L)$. They will also play a role in our structure theory for pro-p algebras of finite rank.

Definition 5.1: Let L be a pro-p algebra. We say that L is **powerful** if

$$L' \subseteq \begin{cases} \overline{L^p}, & \text{if } p \text{ is odd;} \\ \overline{L^4}, & \text{if } p = 2. \end{cases} \qquad ∎$$

The definition of powerful seems more symmetric in the odd and even cases once we realise that L is powerful precisely when $L' \subseteq \overline{D_3(L)}$. Powerful pro-$p$ algebras are clearly the analogues of powerful pro-p groups studied by Lubotzky and Mann in [LM1] and [LM2]. For our present purposes it suffices to study a slightly more general class of pro-p algebras. We shall say that a pro-p algebra L is **weakly powerful** if $\gamma_p(L) + (L')^p \subseteq \overline{L^{p^2}}$. It is easy to see that the powerful and weakly powerful conditions are equivalent when $p = 2$.

PROPOSITION 5.2: *Let L be a weakly powerful pro-p algebra. Then*

1. *L is nilpotent of class at most p,*
2. $D_{p^{i+1}}(L) = (L^{p^i})^p = L^{p^{i+1}} = L^{\{p^{i+1}\}}$, *and*
3. $\dim D_{p^i}(L)/D_{p^{i+1}}(L) \geq \dim D_{p^{i+1}}(L)/D_{p^{i+2}}(L)$ *for each $i \geq 0$.*

Proof: 1. Consider only the case that L is finite: the extension to the general case is standard. By assumption we have $\gamma_p(L) \subseteq L^{p^2}$. But this forces $\gamma_{p+1}(L) \subseteq [L^{p^2}, L] \subseteq \gamma_{p^2+1}(L) = 0$ since L is nilpotent.

2. Let us remark first that it is a consequence of the definition of a restricted Lie algebra that $(x+y)^p \equiv x^p + y^p$ modulo $\gamma_p\langle x, y\rangle$ for all x, y in L. We now

claim that $L^{p^2} = (L^p)^{\{p\}}$. Indeed, suppose $x, y \in L$. Then

$$x^{p^2} + y^{p^2} = (x^p)^p + (y^p)^p \equiv (x^p + y^p)^p \bmod \gamma_p\langle x^p, y^p\rangle.$$

But $\gamma_p\langle x^p, y^p\rangle \subseteq \gamma_{p^2}(L) = 0$, and so the claim follows. Next we prove that $L^p = L^{\{p\}}$. Indeed, $x^p + y^p \equiv (x+y)^p$ modulo $(L^p)^{\{p\}}$ since $\gamma_p(L) \subseteq L^{p^2} = (L^p)^{\{p\}}$. Hence $x^p + y^p = (x+y)^p + z^p$ for some $z \in L^p$. But then $x^p + y^p = ((x+y)+z)^p$, as claimed, since $[L, L^p] \subseteq \gamma_{p+1}(L) = 0$. The case $i = 0$ is now immediate. The case $i \geq 1$ follows by induction and the identity $D_{p^{i+1}}(L) = D_{p^i}(L)^p + \gamma_{p^{i+1}}(L)$.

3. Let $x \in D_{p^i}(L)$ and $y \in D_{p^{i+1}}(L)$. Then $(x+y)^p \equiv x^p + y^p \equiv x^p$ modulo $D_{p^{i+2}}(L)$ because $\gamma_p\langle x, y\rangle \subseteq \gamma_{p^{i+1}+(p-1)p^i}(L) = 0$. Hence the map $\pi_i\colon D_{p^i}(L)/D_{p^{i+1}}(L) \to D_{p^{i+1}}(L)/D_{p^{i+2}}(L)$ induced by the p-map is well defined. In fact π_i is a surjection by part 2 and the fact that $\mathbb{F}$ is perfect. (Notice that π_i need not be linear, unless $\mathbb{F} = \mathbb{F}_p$.) Therefore, $\dim D_{p^i}(L)/D_{p^{i+1}}(L) \geq \dim D_{p^{i+1}}(L)/D_{p^{i+2}}(L)$ since $\mathbb{F}$ is finite. ∎

An argument similar to the proof of part 1 above shows that a powerful pro-p algebra is nilpotent of class at most two.

6. Pro-p algebras of finite rank

Recall from Definition 3.5 that a pro-p algebra L has finite rank if there exists an integer r such that every open p-subalgebra is r-generated. We now list a number of equivalent definitions of finite rank. The proof of the equivalence is straightforward.

Proposition 6.1: *Let r be a positive integer. The following are equivalent for a pro-p algebra L.*

1. $\mathrm{rk}(L) = r$.
2. *The least upper bound on the number of generators required to generate any closed p-subalgebra of L is r.*
3. *The least upper bound on the number of generators required to generate any finitely generated closed p-subalgebra of L is r.*
4. *The least upper bound on $\mathrm{rk}(L/I)$ for all $I \lhd_o L$ is r.*

Theorem 6.2: *Let L be a finitely generated weakly powerful pro-p algebra. Then $\mathrm{rk}(L) = d(L)$.*

Proof: By Proposition 6.1, it suffices to consider the case that L is finite. For every $x \in L$ define its weight by

$$\nu(x) = \begin{cases} \max\{j \mid x \in D_{p^j}(L)\}, & x \neq 0; \\ \infty, & x = 0. \end{cases}$$

For $X \subseteq L$ define

$$\nu(X) = \sum_{x \in X} \nu(x).$$

Given $H \leq L$ we wish to show that $d(H) \leq d(L)$. Assume to the contrary, and among the minimal generating sets of H choose one, X, with $\nu(X)$ maximal. Let $X = \{x_1, \ldots, x_k\}$, and assume without loss that $\nu(x_1) \leq \nu(x_2) \leq \cdots \leq \nu(x_k)$. Using Proposition 5.2 we see that there exists a set $Y = \{y_1, \ldots, y_k\}$ of weight 0 such that $y_i^{p^{\nu(x_i)}} = x_i$. Now $\dim L/\Phi(L) = d(L)$, so Y is a linearly dependent set modulo $\Phi(L)$. Let m be minimal such that y_m lies in the $\mathbb{F}$-span of $\{y_1, \ldots, y_{m-1}\}$ modulo $\Phi(L)$, and put $y_m = \sum_{i=1}^{m-1} \alpha_i y_i + w$, where $w \in \Phi(L)$. Then

$$y_m^p \equiv \sum_{i=1}^{m-1} \alpha_i^p y_i^p + w^p \bmod \gamma_p(L).$$

But $\gamma_p(L) + \Phi(L)^p \subseteq L^{p^2} = D_{p^2}(L)$, so that $y_m^p \equiv \sum_{i=1}^{m-1} \alpha_i^p y_i^p \bmod D_{p^2}(L)$. Proceeding inductively yields

$$x_m = y_m^{p^{\nu(x_m)}} = \sum_{i=1}^{m-1} \alpha_i^{p^{\nu(x_m)}} y_i^{p^{\nu(x_m)}} \bmod D_{p^{\nu(x_m)+1}}(L).$$

However $\nu(x_m) \geq \nu(x_i)$ for $m \geq i$ so

$$x_m = \sum_{i=1}^{m-1} \alpha_i^{p^{\nu(x_m)}} x_i^{p^{\nu(x_m)-\nu(x_i)}} \bmod D_{p^{\nu(x_m)+1}}(L).$$

Thus $x_m = x + z$ for some $x \in \langle x_1, \ldots, x_{m-1} \rangle$ and $z \in L^{p^{\nu(x_m)+1}}$. Hence $\langle x_1, \ldots, x_{m-1}, z, x_{m+1}, \ldots, x_k \rangle = H$. However, this generating set has strictly greater weight than X, contradicting our choice of X. ∎

Observe that, in particular, finitely generated abelian pro-p algebras A satisfy the property $\mathrm{rk}(A) = d(A)$.

We now show that in odd characteristic any pro-p algebra of finite rank contains a large powerful open p-ideal.

Definition 6.3: Let L be a finitely generated pro-p algebra. For each positive r denote by $\mathrm{Tr}_0(r,\mathbb{F})$ the p-subalgebra of $\mathrm{sl}(r,\mathbb{F})$ consisting of strictly upper-triangular matrices. Put $V_r(L)=\bigcap_\psi \ker\psi$, where the intersection is over the set of all restricted representations $\psi: L\to \mathrm{Tr}_0(r,\mathbb{F})$. ∎

Notice that $D_r(L)\subseteq V_r(L)$. Indeed, put $T=\mathrm{Tr}_0(r,\mathbb{F})$. Certainly $\psi(D_r(L))\subseteq D_r(T)$, so it suffices to show that $\gamma_i(T)^{p^j}=0$ whenever $ip^j\geq r$. Now if we let $t\in\gamma_i(T)$ then the first $(i-1)$ diagonals above the main diagonal of t are zero, so $t^{\lceil \frac{r}{i}\rceil}=0$. Because the p-map in T is simply exponentiation by p, we have $t^{p^j}=0$, as claimed.

PROPOSITION 6.4: *Suppose that p is odd, r is a positive integer, and L is a finitely generated pro-p algebra. Then every r-generated open p-ideal N of L contained in $V_r(L)$ satisfies $[N,V_r(L)]\subseteq N^p$.*

Proof: Let us start with the case that L is finite. We begin by proving the following.

CLAIM: Suppose that $N\leq V$ are p-ideals of L. Then either $[N,V]\leq D_3(N)$ or there exists a p-ideal J of L of codimension 1 in $D_3(N)+[N,V]$ such that $D_3(N)+[N,V,L]\leq J< D_3(N)+[N,V]$.

To prove the claim, first observe that $D_3(N)=N^p+\gamma_3(N)$ because $p\geq 3$. Now let $M=D_3(N)+[N,V]$, and assume to the contrary that $M>D_3(N)$. It is easy to see that both M and $D_3(N)$ are p-ideals of L. By the nilpotence of $L/D_3(N)$, we find that

$$M>D_3(N)+[M,L]=D_3(N)+[N,V,L].$$

Now put $J_1=D_3(N)+[N,V,L]$. Then J_1 is a p-ideal of L, and M/J_1 is central in L/J_1 and of exponent p. Thus there exists a p-ideal J of L with $J_1\leq J<M$ and $\dim M/J=1$, as required.

Now put $V=V_r(L)$ and assume that N is an r-generated p-ideal of L contained in V. By induction on the dimension of N, we now prove that $[N,V]\subseteq D_3(N)$. Using the claim above we may assume to the contrary that there exists a p-ideal J of codimension 1 in $M=D_3(N)+[N,V]$, and factor by J to assume that $N^p=0$ and $\dim[N,V]=1$. It now follows that there exists a p-ideal K of L of codimension 1 in N and containing $[N,V]$. Indeed, the nilpotence of L implies that $N>[N,L]\geq[N,V]$, and the existence of K follows from the centrality of $N/[N,L]$ in $L/[N,L]$. Observe next that since $N/[N,V]$ is abelian of exponent p, we must have $d(K/[N,V]) = d(N/[N,V]) - 1 \leq r-1$. Furthermore, $\dim[N,V] = 1$, and so $d(K) \leq r$. Now using the induction hypothesis we find that $[K,V] \leq K^p = 0$. In particular, K is central in N and N/K is 1-dimensional. It follows that N is abelian and hence has dimension at most r. Thus $\mathrm{ad}(V)$ acts trivially on N. This leads us to the desired contradiction.

Therefore, in the finite case, we have $[N,V] \subseteq D_3(N)$. In particular it follows that N is powerful and thus $[N,V] \subseteq N^p$.

Finally, consider the general case. From the finite case we know that

$$[N, V_r(L)] + I/I \subseteq [N+I/I, V_r(L/I)] \subseteq (N+I)^p/I = N^p + I/I$$

for all $I \triangleleft_o L$. But then $[N, V_r(L)] \subseteq \overline{N^p} = N^p$ by Proposition 3.1 and Proposition 5.2. ∎

We remark that Proposition 6.4 has an analogue in characteristic 2, namely that if N is an r-generated open p-ideal of L and $N \leq V_r(L)^2$ then $[N, V_r(L)^2] \subseteq N^4$. In particular, if L has rank r then $V_r(L)^2$ is powerful. However we do not require this result, and we therefore omit the proof.

COROLLARY 6.5: *Suppose that p is odd and L is a pro-p algebra of finite rank r. Then $V_r(L)$ is a powerful open p-ideal of L with codimension at most $r\lceil\log_2 r\rceil$.*

The proof of the corollary follows from Proposition 6.4 and the next lemma.

LEMMA 6.6: *Suppose L is a pro-p algebra of finite rank r. Then $\dim L/D_n(L) \leq r\lceil\log_2 n\rceil$ for each positive integer n.*

Proof: The p-map acts trivially on each of the abelian factors $D_{2^i}(L)/D_{2^{i+1}}(L)$. As L has rank r, each of these factors has dimension at most r. ∎

The following is the analogue of a group-theoretic lemma due to Baer.

LEMMA 6.7: *Let L be a pro-p algebra and $N \triangleleft_c L$. Then*

$$\max\{\mathrm{rk}(N), \mathrm{rk}(L/N)\} \leq \mathrm{rk}(L) \leq \mathrm{rk}(N) + \mathrm{rk}(L/N).$$

In particular the class of pro-p algebras of finite rank is extension closed.

Proof: The first inequality follows from Proposition 6.1. For the second, suppose that $\mathrm{rk}(N) = r$ and $\mathrm{rk}(L/N) = s$. Let H be a d-generated open p-subalgebra of L. We shall show that $d \leq r+s$. By Proposition 3.3 it suffices to show that $d(H+I/I) \leq r+s$ for arbitrary $I \lhd_o L$. Now $\text{rk}(N+I/I) \leq r$ and $(\text{rk}(L/N+I)) \leq s$, so replacing L by L/I it suffices to consider the case that L is $\mathcal{F}_p$. Let $H = \langle h_1, \ldots, h_d \rangle$. Thus we may write each $h_i = f_i(a_1, \ldots, a_s) + l_i$ where $a_1, \ldots, a_s \in H$, f_i is a word in the free Lie p-algebra, and $l_i \in H \cap N$. Moreover there exist $b_1, \ldots, b_r \in H \cap N$ such that $\langle l_1, \ldots, l_d \rangle = \langle b_1, \ldots, b_r \rangle$. Therefore $H = \langle a_1, \ldots, a_s, b_1, \ldots, b_r \rangle$, so $d(H) \leq r+s$. Therefore $\text{rk}(L) \leq r+s$, as required. ∎

We are now ready to state the main results of this section.

THEOREM 6.8: *Let L be a pro-p algebra.*

1. *Suppose that L has finite rank r. Then L is nilpotent of class c such that*

$$c \leq \begin{cases} 2r, & \text{if } p \text{ is odd;} \\ 2r+1, & \text{if } p = 2. \end{cases}$$

It follows that the centre of L is open and satisfies

$$\dim L/\zeta(L) \leq \begin{cases} r\lceil \log_2(2r) \rceil, & \text{if } p \text{ is odd;} \\ r\lceil \log_2(2r+1) \rceil, & \text{if } p = 2. \end{cases}$$

2. *Conversely, assume that L is d-generated and $\dim L/\zeta(L) = n < \infty$. Then $\text{rk}(L) \leq p^n(d-1)+n+1$.*

Proof: 1. First assume that $p=2$ and consider $D_{r+1}(L)$. Notice that $D_{r+1}(L)$ is finitely generated by Theorem 3.4. Therefore by Corollary 4.5 we find that $\Phi(D_{r+1}(L)) = D_{r+1}(L)^2 + D_{r+1}(L)' \subseteq D_{2r+2}(L)$. Then by assumption

$$\dim D_{r+1}(L)/\dim D_{2r+2}(L) \leq d(D_{r+1}(L)) \leq r.$$

Therefore $\gamma_{2r+1}(L) = [D_{r+1}(L), {}_rL] \subseteq D_{2r+2}(L)$, and hence $\gamma_{2r+2}(L) = \gamma_{2r+3}(L) = 0$.

Now assume that p is odd, and consider $D_r(L)$. Then by Corollary 6.5, $D_r(L)$ is powerful. Thus $\Phi(D_r(L)) = D_r(L)^p \subseteq D_{pr}(L)$, and so

$$\dim D_r(L)/D_{pr}(L) \leq d(D_r(L)) \leq r.$$

But then $\gamma_{2r+1}(L) = 0$ as above. The result now follows by Lemma 6.6.

2. Observe first that $\zeta(L)$ is always closed, so that in this case $\zeta(L)$ is open. Hence by Theorems 3.4 and 6.2 and Lemma 6.7 we have

$$\text{rk}(L) \leq \text{rk}(\zeta(L)) + \text{rk}(L/\zeta(L)) \leq d(\zeta(L)) + n \leq p^n(d-1)+1+n. \qquad ∎$$

Let us remark that there is another way to view a converse to part 1 of the theorem.

PROPOSITION 6.9: *Suppose L is a d-generated pro-p algebra that is nilpotent of class c. Then* $\mathrm{rk}(L) \leq p^t(d-1)+t+1$, *where* $t = \sum_{i=1}^{s} d^i \lceil \log_p(s/i) \rceil$ *and* $s = \lceil \frac{c+1}{2} \rceil$.

Proof: Because $\gamma_{c+1}(L) = 0$, $D_s(L)$ is abelian. Observe that $\dim L/D_s(L)$ is bounded by a function t which depends only on d, s, and p. Indeed, $t = \sum_{i=1}^{s} d^i \lceil \log_p(s/i) \rceil$ is a crude upper bound of $\dim L/D_s(L)$, for if L is generated by $\{x_1, \ldots x_d\}$ then L is spanned, modulo $D_s(L)$, by all products of the form $[x_{k_1}, x_{k_2}, \ldots, x_{k_i}]^{p^j}$ where $ip^j < s$. For each value of i there are d^i choices for the i-tuple $(x_{k_1}, x_{k_2}, \ldots, x_{k_i})$, and $\lceil \log_p(s/i) \rceil$ values for j in the range $0 \leq j \leq \lceil \log_p(s/i) \rceil - 1$. By Theorem 3.4, it follows that $d(D_s(L)) \leq p^t(d-1)+1$. Thus by Lemma 6.7 and Theorem 6.2 we have

$$\mathrm{rk}(L) \leq \mathrm{rk}(D_s(L)) + \mathrm{rk}(L/D_s(L)) \leq d(D_s(L)) + \dim L/D_s(L) \leq p^t(d-1)+t+1.$$

This is the required bound on the rank of L. ∎

COROLLARY 6.10: *Let L be a pro-p algebra. Then the following are equivalent:*

1. *L has finite rank;*
2. *L is finitely generated and contains a powerful open p-ideal;*
3. *L is finitely generated and the centre of L is open; and*
4. *L is finitely generated and nilpotent.*

7. Uniform pro-p algebras

For the sake of brevity, let us put $P_i(L) = D_{p^i}(L)$.

Definition 7.1: A pro-p algebra L is said to be **uniform** if it is finitely generated and powerful, and for all i we have $\dim P_i(L)/P_{i+1}(L) = d(L)$. ∎

PROPOSITION 7.2: *If L is a finitely generated powerful pro-p algebra then $P_k(L)$ is uniform for all sufficiently large k. In particular, every pro-p of finite rank contains a uniform open central p-ideal.*

Proof: Write $d_i = \dim P_i(L)/P_{i+1}(L)$. Then by Proposition 5.2, we have $d_i \geq d_{i+1}$ for all i. Thus there exists a positive integer k for which $d_m = d_k$ whenever

$m \geq k$. Now, also by Proposition 5.2, we find that $P_i(P_k(L)) = P_{k+i-1}(L)$. Hence $P_k(L)$ is uniform. The remaining part of the proposition now follows from Corollary 6.10. ∎

PROPOSITION 7.3: *The following are equivalent for a finitely generated powerful pro-p algebra.*

1. *L is uniform.*
2. *$d(P_i(L)) = d(L)$ for all positive i.*
3. *$d(L) = d(H)$ for every powerful open p-subalgebra H of L.*

Proof: This follows from Proposition 5.2 and Theorem 6.2. ∎

COROLLARY 7.4:

1. *If L is a pro-p algebra of finite rank, then L contains an open p-ideal H such that every open p-ideal of L contained in H is uniform.*
2. *If H and K are open uniform p-subalgebras of some pro-p algebra L then $d(H) = d(K)$.*

It is now possible to make the following definition.

Definition 7.5: Let L be a pro-p algebra of finite rank. Then the **uniform dimension** of L is $\text{udim}(L) = d(H)$, where H is any open uniform p-subalgebra of L. ∎

For the remainder of this section we will study uniform pro-p algebras with a view to determining a concrete structure theory for arbitrary pro-p algebras of finite rank.

PROPOSITION 7.6: *Let L be a finitely generated powerful pro-p algebra. Then L is uniform if and only if its p-map is non-singular.*

Proof: If L is uniform then by Proposition 5.2 the p-map induces a bijection π_i: $P_i(L)/P_{i+1}(L) \to P_{i+1}(L)/P_{i+2}(L)$ for all i. Assume that there exists a non-zero element y in L which satisfies $y^p = 0$. Choose i so that y is contained in $P_i(L)$ but not in $P_{i+1}(L)$. But then π_i is not injective.

Conversely assume that L is not uniform, so that there exists a positive integer i for which π_i is not injective. Thus there exists an element y lying in $P_i(L)$ but not in $P_{i+1}(L)$ such that $y^p \in P_{i+2}(L)$. But by Proposition 5.2, $P_{i+2}(L) =$

$P_{i+1}(L)^{\{p\}}$, so there exists $z \in P_{i+1}(L)$ with $z^p = y^p$. Hence $x = y - z \neq 0$ but $x^p = 0$ since z is central. ∎

Next we shall define an action of the formal power series ring $\mathbb{F}[[t]]$ on elements of a pro-p algebra L. We consider $\mathbb{F}[[t]]$ as a pro-p algebra by defining the Lie product to be trivial, and defining the p-map by

$$\Big(\sum_{i\geq 0} \alpha_i t^i\Big)^{[p]} := \sum_{i\geq 0} \alpha_i^p t^{i+1}.$$

In the case that the field is $\mathbb{F}_p$ this is simply the pro-p algebra described in Example 2.4.

Let $h \in L$. If $a(t) = a_0 + a_1 t + \cdots + a_m t^m \in \mathbb{F}[t]$ then define

$$h^{a(t)} = a_0 h + a_1 h^p + \cdots + a_m h^{p^m}.$$

Let $(a_i(t))_{i\geq 0}$ and $(b_i(t))_{i\geq 0}$ be sequences of polynomials converging to the same limit in $\mathbb{F}[[t]]$. We shall show that the sequences $(h^{a_i(t)})_{i\geq 0}$ and $(h^{b_i(t)})_{i\geq 0}$ converge in L with the same limit. Fix $I \triangleleft_o L$, and let n be the least integer such that $L^{p^n} \subseteq I$. For all sufficiently large i and j we have $a_i(t) \equiv a_j(t) \bmod t^n$. Thus $h^{a_i(t)} \equiv h^{a_j(t)} \bmod I$. Therefore $(h^{a_i(t)})_{i\geq 0}$ is Cauchy and by Proposition 3.1 converges with limit h_a, say. Similarly $(h^{b_i(t)})_{i\geq 0}$ converges with limit h_b. Now given I we may choose j sufficiently large that $b_j(t) \equiv a_j(t) \bmod t^n$, $h^{a_j(t)} \equiv h_a \bmod I$, and $h^{b_j(t)} \equiv h_b \bmod I$. Therefore $h_a - h_b \equiv h^{a_j(t)} - h^{b_j(t)} \equiv h^{a_j(t)-b_j(t)} \equiv 0 \bmod I$. As this occurs for arbitrary I we have $h_a = h_b$, as required. Therefore we may make the following definition. For $\lambda(t) \in \mathbb{F}[[t]]$ define $h^{\lambda(t)} = \lim_{i\to\infty} h^{a_i(t)}$, where $(a_i(t))_{i\geq 0}$ is any sequence of polynomials converging to $\lambda(t)$. It follows easily from the definition of the action that if $\alpha(t) = \sum a_i t^i$ and $\beta(t) = \sum b_i t^i$ lie in $\mathbb{F}[[t]]$ then

$$h^{\alpha(t)+\beta(t)} = h^{\alpha(t)} + h^{\beta(t)} \quad\text{and}\quad h^{(\alpha(t)^{[p]})} = (h^{\alpha(t)})^p$$

We now claim that if $L = \overline{\langle a_1, \ldots, a_d\rangle}$ is powerful then $L = \overline{\langle a_1\rangle} + \cdots + \overline{\langle a_d\rangle}$. The proof is in the case that L is $\mathcal{F}_p$: the extension to infinite L is standard. Observe that $L/L^p = \langle a_1\rangle + \cdots + \langle a_d\rangle + L^p/L^p$. Applying the surjection induced by the p-map we find that L^p/L^{p^2} is generated by $\{a_1^p + L^{p^2}, \ldots, a_d^p + L^{p^2}\}$. Now because $L^{p^2} = \Phi(L^p)$, we find that $L^p = \langle a_1^p, \ldots, a_d^p\rangle$. The claim now follows from the fact that L^p is central. We are now ready to establish the following:

PROPOSITION 7.7: *Let L be a d-generated uniform abelian pro-p algebra. Then $L \cong \bigoplus_1^d \mathbb{F}[[t]]$ as topological Lie p-algebras.*

Proof: Let $\{a_1, \dots a_d\}$ be a generating set for L. Then

$$L = \overline{\langle a_1 \rangle} + \cdots + \overline{\langle a_d \rangle}.$$

Therefore we may express each element of L in the form $a_1^{\lambda_1} + \cdots + a_d^{\lambda_d}$ with the $\lambda_i \in \mathbb{F}[[t]]$. In fact we shall show that this expression is unique. In light of the discussion above it suffices to prove the expression for 0 is unique. Assume there exist $\lambda_1, \dots, \lambda_d$ not all zero such that $a_1^{\lambda_1} + \cdots + a_d^{\lambda_d} = 0$. Now $\dim L/L^p = d$ so considering the exponents modulo t we find each $\lambda_i \equiv 0 \bmod t$. Therefore there exist $\kappa_1, \dots, \kappa_d$ such that $\kappa_i^{[p]} = \lambda_i$ for all i, and so $(a_1^{\kappa_1} + \cdots + a_d^{\kappa_d})^p = 0$. However the preceding proposition shows that the p-map is non-singular, so $a_1^{\kappa_1} + \cdots + a_d^{\kappa_d} = 0$. Proceeding inductively we arrive at the desired contradiction. Consequently we obtain a bijection $\varphi: L \to \bigoplus_1^d \mathbb{F}[[t]]$ given by $\varphi(l) = (\lambda_1, \dots, \lambda_d)$ when $l = \sum a_i^{\lambda_i}$. It is straightforward to verify that this is a homomorphism of restricted Lie algebras. From Theorem 4.6 it is easy to see that every homomorphism from a finitely generated pro-p algebra onto another is continuous, and thus φ is continuous. Therefore since L and $\bigoplus_1^d \mathbb{F}[[t]]$ are compact and Hausdorff, they are homeomorphic. ∎

The following result closely describes the structure of pro-p algebras of finite rank.

THEOREM 7.8: *Let L be a pro-p algebra of finite rank. Put $s = \text{udim}(L)$. Then there exists a central open p-ideal U of L such that $U \cong \bigoplus_1^s \mathbb{F}[[t]]$.*

Proof: It follows from Proposition 7.2 and Corollary 7.4 that L contains a uniform open central p-ideal U of rank s. By Proposition 7.7, U is isomorphic to $\bigoplus_1^s \mathbb{F}[[t]]$ as topological restricted Lie algebras, as required. ∎

Note that it is not possible to bound the codimension of U in L in terms of $\text{rk}(L)$.

8. Pro-p groups of finite rank

In this final section we illustrate a concrete connection between the study of pro-p algebras and pro-p groups. Recall the graded restricted Lie $\mathbb{F}_p$-algebra associated to a group G defined by

$$\mathrm{gr}(G) = \bigoplus_{m\geq 1} D_m(G)/D_{m+1}(G).$$

Here $D_m(G)$ represents the mth **dimension subgroup** of G given by $D_n(G) = G \cap (1 + \Delta(G)^n)$, where $\Delta(G)$ denotes the augmentation ideal of the group ring $\mathbb{F}_p G$. Commutation and exponentiation by p in G induce the restricted Lie structure on $\mathrm{gr}(G)$. In fact we are more interested in its pro-$\mathcal{F}_p$ completion, $\widehat{\mathrm{gr}(G)}$, which we call the **associated pro-p algebra** of G and denote by $\mathcal{L}(G)$.

In [L1], Lazard gave an explicit description for the dimension subgroups of G, namely

$$D_m(G) = \prod_{ip^j \geq m} \gamma_i(G)^{p^j}.$$

The dimension subalgebras of a Lie p-algebra arise in a similar fashion. Indeed, let $\omega(L)$ represent the augmentation ideal of the restricted universal enveloping algebra of L. In [RS] it is shown that setting

$$D_m(L) = L \cap \omega(L)^m$$

for each $m \geq 1$ is consistent with the definition $D_m(L) = \sum_{ip^j \geq m} \gamma_i(L)^{p^j}$ given in Section 4.

In what follows, we shall use freely the notation we have established for pro-p algebras to pro-p groups as well. The reader is referred to [DDMS] for the basic theory about pro-p groups of finite rank.

LEMMA 8.1: *Let G be an arbitrary group. Then for each pair of positive integers m and n, $[D_m(G), D_n(G)] \subseteq \gamma_{m+n}(G)D_{m+n+1}(G)$.*

Proof: We begin with the case $[D_m(G), G] \subseteq \gamma_{m+1}(G)D_{m+2}(G)$. It suffices to show that if $ip^j \geq m$, $x \in \gamma_i(G)$ and $y \in G$, then $[x^{p^j}, y] \subseteq \gamma_{m+1}(G)D_{m+2}(G)$. Let $H = \langle [x,y], x \rangle$. By Hall's collection formula

$$[x^{p^j}, y] \equiv [x,y]^{p^j} \bmod (H')^{p^j} \prod_{k=1}^{j} \gamma_{p^k}(H)^{p^{j-k}}.$$

Without loss we may assume that $j \geq 1$. Since

$$[x,y]^{p^j} \in \gamma_{i+1}(G)^{p^j} \subseteq D_{m+p^j}(G),$$

$$(H')^{p^j} \subseteq \gamma_{2i+1}(G)^{p^j} \subseteq D_{2m+p^j}(G),$$
$$\gamma_{p^j}(H) \subseteq \gamma_{ip^j+1}(G) \subseteq \gamma_{m+1}(G), \quad \text{and}$$
$$\gamma_{p^k}(H)^{p^{j-k}} \subseteq \gamma_{ip^k+1}(G)^{p^{j-k}} \subseteq D_{m+p}(G) \quad \text{if} k \neq j,$$

the proof of this case follows.

Now $[D_m(G), \gamma_i(G)] \subseteq [D_m(G),_i G] \subseteq \gamma_{m+i}(G)D_{m+2i}(G)$. The result follows from another application of Hall's collection formula. ∎

See Riley, [R], for stronger results of this type.

LEMMA 8.2: *Let G be a finitely generated pro-p group. Then*

$$\mathcal{L}(G) = \prod_{i\geq 1} D_i(G)/D_{i+1}(G),$$

and for each positive integer m

$$D_m(\mathcal{L}(G)) = \prod_{i\geq m} D_i(G)/D_{i+1}(G).$$

Proof: Because G is finitely generated, $G/\Phi(G) = D_1(G)/D_2(G)$ is finite. Therefore $\operatorname{gr}(G)$ is finitely generated as a restricted Lie algebra. Put $E_m = \bigoplus_{i\geq m} D_i(G)/D_{i+1}(G)$ for each m. Then each factor $\operatorname{gr}(G)/E_m$ lies in $\mathcal{F}_p$ because $G/D_m(G)$ is a finite p-group. Now, using Theorem 4.6 and an 'unravelling' map similar to that employed in Example 2.4, it is not difficult to check that

$$\mathcal{L}(G) = \varprojlim_m \operatorname{gr}(G)/E_m \cong \prod_{m\geq 1} D_m(G)/D_{m+1}(G).$$

Therefore using Lemma 8.1 we see by induction that

$$\gamma_i(\mathcal{L}(G)) = \prod_{k\geq i} \frac{\gamma_k(G)D_{k+1}(G)}{D_{k+1}(G)}$$

for each $i \geq 2$. The result now follows using the formulae for dimension subgroups and dimension subalgebras given above. ∎

For the remainder of the article let us abbreviate $\mathcal{L}(G)$ by L, and for each $i \geq 1$ put $d_i = \log_p |D_i(G)/D_{i+1}(G)|$. From the preceding lemma it follows that $d(G) = d_1 = d(L)$ and $d_i = \dim D_i(L)/D_{i+1}(L)$ for each $i \geq 2$. Also observe that if G is a finitely generated pro-p group then G is powerful if and only if L is abelian. Furthermore, in this situation $\operatorname{rk}(G) = \operatorname{rk}(L)$ because $\operatorname{rk}(L) = d(L)$

by Theorem 6.2 and $\mathrm{rk}(G) = d(G)$ by [DDMS, Theorem 3.8]. This motivates us to consider the relationship between $\mathrm{rk}(G)$ and $\mathrm{rk}(L)$ for general pro-p groups of finite rank.

To simplify the statements of the following results, we set $\epsilon = 0$ if p is odd and $\epsilon = 1$ if $p = 2$.

LEMMA 8.3: *Suppose that G is a pro-p group with finite rank r. Then*

1. *$d_i = 0$ for some $i \leq 2r + \epsilon$,*
2. *L is nilpotent of class at most $2r - 1 + \epsilon$,*
3. *$D_{r+\epsilon}(L)$ is abelian, and*
4. *$d(D_t(L)) = d(D_t(G)) \leq r$ for all $t \geq r + \epsilon$.*

Proof: 1. Since $d(D_{r+1}(G)) = \log_p |D_{r+1}(G)/\Phi(D_{r+1}(G))| \leq r$ and

$$\Phi(D_{r+1}(G)) = D_{r+1}(G)^p D_{r+1}(G)' \subseteq D_{2r+2}(G),$$

it follows that $\log_p |D_{r+1}(G)/D_{2r+2}(G)| \leq r$. Hence $d_{r+1} + \cdots + d_{2r+1} \leq r$, so that $d_i = 0$ for some $i \leq 2r + 1$. When p is odd, $D_r(G)$ is powerful according to theorem due to Lubotzky and Mann, [DDMS, Proposition 3.9]. Therefore $\Phi(D_r(G)) = \overline{D_r(G)^p} \subseteq D_{2r+1}(G)$. Counting again we find that $d_{2r} = 0$.

2. This follows at once from part 1 and Lemma 8.2.

3. This is a corollary of part 2.

4. Let us prove only the case $t = r + \epsilon$ for the general case is similar. Suppose first that p is odd. Then, as above, $D_r(G)$ is powerful. In other words, $\Phi(D_r(G)) = D_r(G)^p$. We claim that this implies that $\Phi(D_r(G)) = D_{pr}(G)$. Indeed, $d(D_r(L)) \leq r$ forces

$$\gamma_{2r}(G) \subseteq [D_r(G), {}_rG] \subseteq \Phi(D_r(G)).$$

Thus $D_{pr}(G) = D_r(G)^p \gamma_{pr}(G) = D_r(G)^p = \Phi(D_r(G))$. Since $\gamma_{2r}(L) = 0$, clearly $\Phi(D_r(L)) = D_r(L)^p = D_{pr}(L)$. Hence $d(D_r(L)) = d_r + \cdots + d_{pr-1} = d(D_r(G))$, as required.

Now assume that $p = 2$. Then, as above, $d(D_{r+1}(G)) \leq r$ forces $\gamma_{2r+1}(G) \subseteq \Phi(D_{r+1}(G))$. But $D_{r+1}(G)/D_{r+1}(G)^2$ is abelian, so that

$$\Phi(D_{r+1}(G)) = D_{r+1}(G)^2 = D_{2r+2}(G).$$

Because $\gamma_{2r+1}(L) = 0$, certainly $\Phi(D_{r+1}(L)) = D_{r+1}(L)^2 = D_{2r+2}(L)$. Thus $d(D_{r+1}(L)) = d_{r+1} + \cdots + d_{2r+1} = d(D_{r+1}(G))$. ∎

LEMMA 8.4: *Suppose that G is a pro-p group such that L finite rank s. Then the following statements hold.*

1. $d_i = 0$ *for some* $i \leq 2s + \epsilon$.
2. L *is nilpotent of class at most* $2s - 1 + \epsilon$.
3. $d(D_t(G)) = d(D_t(L)) \leq s$ *for all* $t \geq s + \epsilon$.
4. $D_{s+\epsilon(s+2)}(G)$ *is powerful.*

Proof: The proofs of parts 1 and 2 are essentially the same as those of Lemma 8.3. For part 3, let us prove only the case that $t = s + \epsilon$. Suppose first that p is odd. Then, by part 2, $\Phi(D_s(L)) = D_s(L)^p = D_{ps}(L)$. By part 1 we have $\gamma_{2s}(G) \subseteq D_{2s+1}(G)$, and hence $\gamma_m(G) \subseteq D_{m+1}(G)$ for all $m \geq 2s$. But then

$$D_{ps}(G) = D_s(G)^p \gamma_{ps}(G) = D_s(G)^p D_{ps+1}(G) = \bigcap_{m \geq ps} D_s(G)^p D_m(G) = \overline{D_s(G)^p}.$$

Therefore $D_{ps}(G) \subseteq \Phi(D_s(G))$, and consequently $d(D_s(G)) \leq d_s + \cdots + d_{ps-1} = d(D_s(L)) \leq s$. It follows that $D_s(G)$ is powerful. This implies that $\Phi(D_s(G)) = D_{ps}(G)$ and in turn that $d(D_s(G)) = d(D_s(L))$.

Finally, let $p = 2$. Arguing as above yields $\Phi(D_{s+1}(L)) = D_{2s+2}(L)$ and $D_{2s+2}(G) = D_{s+1}(G)^2 \gamma_{2s+2}(G) = \overline{D_{s+1}(G)^2} = \Phi(D_{s+1}(G))$. It follows that $d(D_{s+1}(G)) = d(D_{s+1}(L))$. This finishes the proof of part 3.

Taking $t = 2s + 2$ in part 3 we find that $d(D_{2s+2}(G)) \leq s$. Recall that $D_{2s+2}(G) = \overline{D_{s+1}(G)^2}$. It now follows by the characteristic 2 version of Lubotzky and Mann's theorem that $D_{2s+2}(G)$ is powerful. This proves part 4. ∎

We are now ready for our main results about the close relationship between the rank of a pro-p group and the rank of its associated pro-p algebra.

THEOREM 8.5: *Let G be a pro-p group. Write* $r = \mathrm{rk}(G)$ *and* $s = \mathrm{rk}(L)$. *Then*

1. $\mathrm{rk}(G)$ *is finite if and only if* $\mathrm{rk}(L)$ *is finite; moreover*
2. $\mathrm{rk}(G) \leq s + s\lceil \log_2(s + \epsilon(s+2)) \rceil$ *and* $\mathrm{rk}(L) \leq r + r\lceil \log_2(r + \epsilon) \rceil$; *and*
3. $\mathrm{udim}(G) = \mathrm{udim}(L)$.

Proof: 1. From Lemmas 8.3 and 8.4 we see that if G has finite rank then L has an open powerful p-subalgebra, and conversely. The result now follows from Corollary 6.10 and its group-theoretic counterpart [DDMS, Theorem 3.13].

2. We can assume from part 1 that r and s are finite. Suppose that p is odd. Then by [DDMS, p. 61] we have $\mathrm{rk}(G) \leq \mathrm{rk}(D_s(G)) + \mathrm{rk}(G/D_s(G))$. But

$D_s(G)$ is powerful by part 4 of Lemma 8.4, so that $\mathrm{rk}(D_s(G)) = d(D_s(G)) \leq s$ by [DDMS, Theorem 3.8] and part 3 of Lemma 8.4. Also

$$\mathrm{rk}(G/D_s(G)) \leq \log_p |G/D_s(G)| = \dim L/D_s(L) \leq s\lceil \log_2 s \rceil$$

by Lemma 6.6. Therefore $\mathrm{rk}(G) \leq s + s\lceil \log_2 s \rceil$, as required. On the other hand, by Lemma 6.7 we have $\mathrm{rk}(L) \leq \mathrm{rk}(D_r(L)) + \mathrm{rk}(L/D_r(L))$. Since $D_r(L)$ is abelian by Lemma 8.3, we have $\mathrm{rk}(D_r(L)) = d(D_r(L)) \leq r$ by the same Lemma and Theorem 6.2. Also

$$\mathrm{rk}(L/D_r(L)) \leq \dim L/D_r(L) = \log_p |G/D_r(G)| \leq r\lceil \log_2 r \rceil,$$

by the group-theoretic analogue of Lemma 6.6. Thus $\mathrm{rk}(L) \leq r + r\lceil \log_2 r \rceil$.

Now suppose that $p = 2$. Because $D_{2s+2}(G)$ is powerful by Lemma 8.4, similar arguments to those above yield

$$\mathrm{rk}(G) \leq \mathrm{rk}(D_{2s+2}(G)) + \mathrm{rk}(G/D_{2s+2}(G)) \leq s + s\lceil \log_2(2s+2) \rceil.$$

On the other hand, since $D_{r+1}(L)$ is abelian by Lemma 8.3, it follows that

$$\mathrm{rk}(L) \leq \mathrm{rk}(D_{r+1}(L)) + \mathrm{rk}(L/D_{r+1}(L)) \leq r + r\lceil \log_2(r+1) \rceil.$$

3. By [DDMS, Corollary 4.3] and Proposition 7.2 we can choose m so large that both $D_m(G)$ and $D_m(L)$ are uniform. Therefore by Lemma 8.3 we have $\mathrm{udim}(G) = d(D_m(G)) = d(D_m(L)) = \mathrm{udim}(L)$. ∎

Lubotzky and Mann proved in [LM2] that if G is a pro-p group of finite rank then it is p-adic analytic of dimension $\mathrm{udim}(G)$. Thus $\mathrm{udim}(L)$ is the dimension of G.

As an application we obtain a quantative version of a result due to Lazard, [L2], and Shalev, [S]. See also Riley, [R].

COROLLARY 8.6: *Let G be a pro-p group.*

1. *If G has finite rank r then $\mathrm{gr}(G)$ is nilpotent of class at most $2r - 1 + \epsilon$.*
2. *Conversely, assume that $d = d(G)$ and $\mathrm{gr}(G)$ is nilpotent of class c. Then G has finite rank bounded by a function of d, c and p only. Namely, $\mathrm{rk}(G) \leq s + s\lceil \log_2(s + \epsilon(s+2)) \rceil$, where $s = p^t(d-1)+t+1$ and $t = \sum_{i=1}^{c} d^i \lceil \log_p(c/i) \rceil$.*

Proof: The first part follows from Lemma 8.3. For the second part put $L = \mathcal{L}(G)$ as before. Then $d = d(L)$ and $\gamma_{c+1}(L) = 0$. Hence by Proposition 6.9 we find that $\mathrm{rk}(L) \leq p^t(d-1) + t + 1$. The result now follows by Theorem 8.5. ∎

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